Exam Percolation Theory

4 November 2022, 8:30-10:30

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100).

Exercise 1 (20 pts).
The *square root trick* states that if $A_1, \dots, A_n$ are either all increasing or all decreasing events then

$$\max_{i=1,\dots,n} \mathbb{P}(A_i) \geq 1 - \sqrt[n]{1 - \mathbb{P}(A_1 \cup \dots \cup A_n)}.$$

Give a proof of the square root trick.
(Make sure to state very clearly any results from the lecture notes you are using.)

Exercise 2 (20 pts).
We consider the "majority function" $f : \{\pm 1\}^3 \to \{\pm 1\}$, given by

$$f(x) = \begin{cases} +1 & \text{if } x_1 + x_2 + x_3 > 0, \text{ and} \\ -1 & \text{otherwise.} \end{cases}$$

i) Determine $\mathrm{Inf}_i(f)$ for $i = 1, \dots, 3$.

ii) Determine the Fourier-Walsh decomposition of f.

Exercise 3 (25 pts)
We consider bond percolation on $\mathbb{Z}^d$ with $p \in (0,1)$ atrbitrary. As usual $e_1 = (1, 0, \dots, 0)$ denotes the first standard basis vector.

i) Show that $\mathbb{P}_p(\underline{0} \leftrightsquigarrow (n+m)e_1) \geq \mathbb{P}_p(\underline{0} \leftrightsquigarrow ne_1) \cdot \mathbb{P}_p(\underline{0} \leftrightsquigarrow me_1)$.

ii) Prove Fekete's lemma: if a sequence of nonnegative numbers $(a_n)_n$ that satisfies $a_{n+m} \leq a_n + a_m$ for all $n, m \in \mathbb{N}$, then $\lim_n a_n/n$ exists and equals $\inf_n a_n/n$.

(*Hint:* note that $a_n \leq \lfloor n/k \rfloor \cdot a_k + a_{n-k\cdot\lfloor n/k \rfloor}$ for all n, k.)

iii) Show that

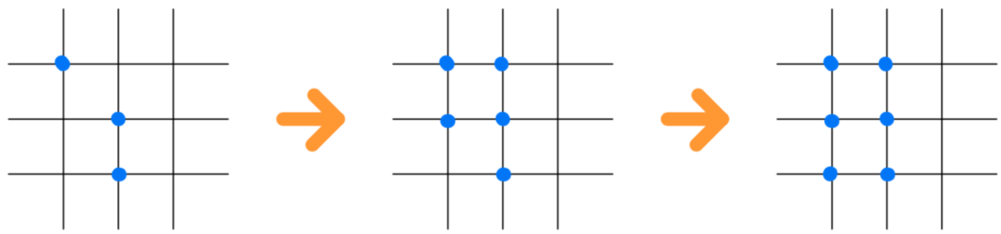
$$\xi(p) := \lim_{n\to\infty} \frac{1}{n} \cdot \ln\left(\frac{1}{\mathbb{P}_p(\underline{0} \leftrightsquigarrow ne_i)}\right),$$

exists.

iii) Show that $\mathbb{P}_p(\underline{0} \leftrightsquigarrow ne_1) \leq e^{-\xi(p)\cdot n}$ for all n.

(See next page)

Exercise 4 (25 pts)
The *bootstrap percolation* process is defined follows. At time $t = 0$ every site of $\mathbb{Z}^2$ is *infected* (with some disease) with probability p, independently of all other sites. At every subsequent time step $t = 1, 2, \ldots$, every vertex that is not yet infected but has at least two infected neighbours also becomes infected. Once infected, sites stay infected forever in this model.



(Example of how the process might evolve starting from the initial configuration on the left.)

We say $z \in \mathbb{Z}^2$ "eventually becomes infected" if there is some (finite, random) t such that z is infected at time t.

The pertinent question is : will every vertex of $\mathbb{Z}^2$ eventually become infected?

If yes we say "bootstrap percolation occurs".

i) Explain why, if all of Λ_n eventually becomes infected, and if on each of the four sides of $\partial\Lambda_{n+1}$ there is at least one infected site at time $t = 0$, then all of Λ_{n+1} will also become infected eventually.

(A short but clear answer suffices here.)

ii) Show that

$$\mathbb{P}_p \left(\begin{array}{c} \text{at least one site in } \Lambda_{n+1} \text{ is never infected, but} \\ \text{all of } \Lambda_n \text{ is eventually infected} \end{array} \right) \leq 4(1-p)^{2n+3}.$$

iii) Show that for every $0 < p \leq 1$ and $\varepsilon > 0$ there exists an $n = n(p, \varepsilon)$ such that

$$\mathbb{P}_p(\text{ bootstrap percolation occurs} \mid \text{all of } \Lambda_n \text{ is infected at time } t = 0\ ) > 1 - \varepsilon.$$

iv) Let us fix some arbitrary n and call a *seed* a translate of Λ_n (i.e. a set of the form $z + \Lambda_n$) all of whose sites are infected at time $t = 0$. Show that for every $0 < p \leq 1$ we have

$$\mathbb{P}_p(\text{there exists at least one seed}) = 1.$$

v) Show that in fact, for all $0 < p \leq 1$:

$$\mathbb{P}_p(\text{bootstrap percolation occurs}) = 1.$$

(*Hint:* It may help to reveal the initial infection in two "stages". More specifically, we could associate i.i.d. uniform random variables $U_z \in [0, 1]$ to each $z \in \mathbb{Z}^2$, and then first "reveal" only which sites satisfy $U_z \leq q$ for some $q < p$.)

(The end)